FIRST AND SECOND SEMESTER B. TECH. (ENGINEERING) DEGREE
MODEL QUESTION PAPER

EN 14 102 - ENGINEERING MATHEMATICS II

Time : 3 Hrs | Max : 100 marks

Part A (Answer any EIGHT questions)

1. Solve : $\left(1+e^{\frac{x}{y}}\right)dx + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}dy = 0$
2. Solve: $(D^2 - 4D + 3)y = sin3x$
3. Find the orthogonal trajectories of the cardioids $r = a(1 - cos\theta)$
4. Find (a) L $(t^2\ sin2t)$ (b) L $\{\frac{1-cost}{t}\}$
5. Find $L^{-1}\left(\frac{2s+1}{(s+2)(s-1)^2}\right)$
6. Find the constants a, b, c so that the following vector is irrotational

$$\bar{F} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$$

7. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11\ and\ xy + yz + zx = 18$ at the point (6,4,3).
8. State Gauss divergence theorem and Stokes theorem.
9. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ show that $grad\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$
10. Using Greens theorem in the plane evaluate $\int_c (xy + y^2)dx + x^2dy$, where c is the closed curve of the region bounded by $y = x\ and\ y = x^2$

$(8 \times 5 = 40\ marks)$

Part B (Answer all questions)

11. (a) (i)Solve $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} = 0$

(ii) Solve $(D^2 - 4D + 4)y = 8x^2e^{2x}sinx$

Or

(b) (i)Using the method of variation of parameters solve $(D^2 + 4)y = \tan 2x$

(ii) Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = logxsin(logx)$

12 (a) (i) Find $L^{-1}\frac{S^2}{(S^2+a^2)(s^2+b^2)}$

(ii) Find $L\left(\int_0^t \frac{e^t sint}{t}dt\right)$

Or

(b) Solve $y'' - 3y' + 2y = 4t + e^{3t}$given that$y(0) = 1$ $and y'(0)$ = - 1

13. (a) (i)Prove that $\nabla^2(\mathrm{r}^{\mathrm{n}}) = \mathrm{n}(\mathrm{n}+1)\mathrm{r}^{\mathrm{n}-2}$ and hence deduce $\nabla^2\left(\frac{1}{r}\right) = 0$

(ii)Prove that $\vec{A}$ = $3y^4z^2\vec{\imath}$ + $4x^3z^2\vec{\jmath}$ -$3x^2y^2\vec{k}$ is a solenoidal vector while

$\vec{B}$ = $2xye^z\vec{\imath}$+$x^2e^z\vec{\jmath}$+$x^2ye^z\vec{k}$ is an irrotational vector

Or

(b) (i) Find the value of the constants λ and μ so that surface λx^2- $\mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ *may inersect orthogonally at the point*$(1,-1,2)$

(ii) Prove that $div\left(\frac{\vec{r}}{r^3}\right) = 0$

14. (a) Use Gauss divergence theorem to evaluate$\iint_s \vec{f}.\hat{n}ds, where$ $\vec{f}$ = $4x\vec{\imath}$-$2y^2\vec{\jmath}$+$z^2\vec{k}$ and

s is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ and $z = 3$

Or

(b) Verify Stokes theorem for a vector field $\vec{A} = (x^2 - y^2)\vec{\imath} + 2xy\vec{\jmath}$ in the rectangular region of the $z = 0$plane bounded by the lines $x = 0, x = a, y = 0$ and $y = b$

Part A

1. Solve : $(1+e^{x/y})dx + (1-\frac{x}{y})e^{x/y}dy = 0$ — (1)

put $x = vy$

$\therefore \frac{dx}{dy} = v + y\frac{dv}{dy}$

$\therefore (1) \Rightarrow \frac{dx}{dy} = -\frac{(1-\frac{x}{y})e^{x/y}}{1+e^{x/y}}$

$v + y\frac{dv}{dy} = \frac{-(1-v)e^{v}}{1+e^{v}}$

$y\frac{dv}{dy} = \frac{-e^{v}+ve^{v}-v-ve^{v}}{1+e^{v}} = \frac{-e^{v}-v}{1+e^{v}} = -\frac{(v+e^{v})}{1+e^{v}}$

$\frac{1+e^{v}}{v+e^{v}}dv = -\frac{dy}{y}$ $\quad\therefore \log(v+e^{v}) = -\log y$

$(v+e^{v})y = c$ $\quad\therefore (\frac{x}{y}+e^{x/y})y = c$

Ans: $x + ye^{x/y} = c$

2. Solve : $(D^2-4D+3)y = \sin 3x$

$m^2-4m+3=0$

$(m-3)(m-1)=0$

$m = 3, 1$ $\quad\therefore C.F = c_1e^{3x}+c_2e^{x}$

$P.I_1 = \frac{1}{D^2-4D+3}\sin 3x = \frac{1}{-9+4D+3}\sin 3x = \frac{1}{4D-6}\sin 3x$

$= \frac{4D+6}{16D^2-36}\sin 3x = \frac{4D+6}{4(4D^2-9)}\sin 3x$

$= \frac{4\cdot 3\cos 3x + 6\sin 3x}{4\cdot(-36-9)} = \frac{12\cos 3x+6\sin 3x}{4\cdot -45}$

$= -\frac{12\cos 3x+6\sin 3x}{180} = -\frac{4\cos 3x+2\sin 3x}{60}$

$\therefore$ Solution $y = c_1e^{3x}+c_2e^{x}-\frac{1}{30}(2\cos 3x+\sin 3x)$

3. The equation of the family of given cardioid is $r = a(1-\cos\theta)$ — ①

Diff ① w.r.t. θ, $\therefore \frac{dr}{d\theta} = a\sin\theta$ — ②

② ÷ ① $\quad \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \cot\theta/2$ — ③

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in ③

$\therefore r\frac{d\theta}{dr} + \cot\theta/2 = 0$ — ④

integrating ④, $\log r = \log c + \log\cos^2\frac{\theta}{2}$

$\therefore r = c(1+\cos\theta)$ is the required O.T

4) (a) $L\{t^2\sin 2t\} = (-1)^2\frac{d^2}{ds^2}\cdot\frac{2}{s^2+4} = \frac{d}{ds}\,-1\cdot\frac{4s}{(s^2+4)^2} = \frac{4(3s^2-4)}{(s^2+4)^3}$

(b) $L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+1}\,ds$

$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty = \frac{1}{2}\left[\log 1 + \log\frac{s^2+1}{s^2}\right]$

$= \frac{1}{2}\log\left[\frac{s^2+1}{s^2}\right]$

5) $L^{-1}\left\{\frac{2s+1}{(s+2)(s-1)^2}\right\}$

Now $\frac{2s+1}{(s+2)(s-1)^2} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$

$2s+1 = A(s-1)^2 + B(s+2)(s-1) + C(s+2)$

$\therefore A = -\frac{1}{3} \quad B = 1/3 \quad C = 1$

$\therefore L^{-1}\left\{\frac{2s+1}{(s+2)(s-1)^2}\right\} = \frac{\cancel{-1/3}}{\cancel{s+2}}\, L^{-1}\left\{\frac{-1/3}{s+2} + \frac{1/3}{s-1} + \frac{1}{(s-1)^2}\right\}$

$= -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{+t} + te^t$

6. $\vec{F} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

Since $\vec{F}$ is irrotational curl $\vec{F} = \vec{0}$

$$\therefore \text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$i(c-1) - j(4-a) + k(b-2) = \vec{0}$$

$\therefore c-1=0 \quad 4-a=0 \quad b-2=0$

$\therefore c=1,\ a=4,\ b=2$

7. normal to the surface $\phi_1 = x^2+y^2-z^2$ is $\nabla\phi_1$

normal to the surface $\phi_2 = xy+yz+zx$ is $\nabla\phi_2$

$\therefore$ angle bet. surfaces $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Now

$$\nabla\phi_1 = 2xi - 2yj - 2zk$$

$$\nabla\phi_2 = i(y+z) + j(x+z) + k(y+x)$$

$$\therefore (\nabla\phi_1)_{(6,4,3)} = 12i - 8j - 6k$$

$$(\nabla\phi_2)_{(6,4,3)} = 7i + 9j + 10k$$

$$\therefore \cos\theta = \frac{(12i-8j-6k)\cdot(7i+9j+10k)}{\sqrt{144+64+36}\cdot\sqrt{49+81+100}} = \frac{-48}{\sqrt{244\times230}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-48}{\sqrt{244\times230}}\right)$$

8. State the Gauss and Stokes theorem.

9. Given $\vec{r} = xi + yj + zk$

$$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + k\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

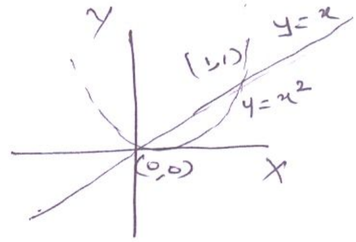
$$= i\,\frac{-1}{r^2}\cdot\frac{\partial r}{\partial x} + j\,\frac{-1}{r^2}\,\frac{\partial r}{\partial y} + k\,\frac{-1}{r^2}\,\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left[i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right] = \frac{-1}{r^2}\left[\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right]$$

$$= -\frac{1}{r^3}\vec{r}$$

10. $\int (xy + y^2)\,dx + x^2\,dy$

Given $y = x$ and $y = x^2$



$$\oint_C M\,dx + N\,dy = \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy$$

$$\frac{\partial N}{\partial x} = 2x \qquad \frac{\partial M}{\partial y} = x + 2y$$

$$\therefore \iint_R \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy = \int_0^1 \int_{x^2}^{x} 2x - x - 2y\;dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} x - 2y\;dy\,dx = \int_0^1 x^4 - x^3\,dx = \left.\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

11.(a)

Part B

(i) $\frac{dy}{dx} = \frac{x - y + 3}{2x - 2y + 5} = \frac{x - y + 3}{2(x-y) + 5}$

put $x - y = t \quad \therefore 1 - \frac{dy}{dx} = \frac{dt}{dx} \quad \therefore \frac{dy}{dx} = 1 - \frac{dt}{dx}$

$1 - \frac{dt}{dx} = \frac{t+3}{2t+5} \quad \therefore -\frac{dt}{dx} = \frac{t+3}{2t+5} - 1 = \frac{t + 3 - 2t - 5}{2t+5} = \frac{-t-2}{2t+5}$

$\frac{2t+5}{t+2}\,dt = +dx$

$\frac{2(t+2) + 1}{t+2}\,dt = +dx$

$\left(2 + \frac{1}{t+2}\right) dt = +dx$

Suing $2t + \log(t+2) = +x + c$

$\therefore 2(x - y) + \log(x - y + 2) = +x + C$

~~3x − 2y + log~~ $x - 2y + \log(x - y + 2) = C$

11.

a (ii) $(D^2-4D+4)y = 8x^2e^{2x}\sin x$

$m^2-4m+4=0 \quad (m-2)^2=0 \quad \therefore m=2,2$

$\therefore C.F = (c_1+c_2x)e^{2x}$

$P.I = \dfrac{1}{D^2-4D+4} 8x^2e^{2x}\sin x$

$= 8.e^{2x}\dfrac{1}{(D+2)^2-4(D+2)+4}x^2\sin x = 8.e^{2x}\dfrac{1}{D^2+4D+4-4D-8+4}x^2\sin x$

$= 8.e^{2x}.\dfrac{1}{D^2}x^2\sin x = 8e^{2x}.\dfrac{1}{D}\left[\int x^2\sin x\,dx\right]$

$= 8e^{2x}\left[-x^2\sin x - 4x\cos x + 6\sin x\right]$

$\therefore$ solution $y=(c_1+c_2x)e^{2x}+8e^{2x}(-x^2\sin x - 4x\cos x+6\sin x)$

or

11 b(i) $(D^2+4)y = \tan 2x$

$m^2+4=0 \quad m^2=-4 \quad \therefore m=\pm 2i \quad \therefore C.F = c_1\cos 2x + c_2\sin 2x$

$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

$P.I = -y_1\int \dfrac{y_2X}{W}dx + y_2\int\dfrac{y_1X}{W}dx$

$= -\cos 2x\int \dfrac{\sin 2x.\tan 2x}{2} + \sin 2x\int\dfrac{\cos 2x\tan 2x}{2}dx$

$= -\dfrac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$

$\therefore$ solution $y = c_1\cos 2x + c_2\sin 2x - \dfrac{1}{4}\cos 2x\log(\sec 2x+\tan 2x)$

b (ii) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y = \log x\sin(\log x)$ —— ①

put $x=e^z \quad \therefore z=\log x \quad \therefore x\dfrac{dy}{dx}=Dy \quad x^2\dfrac{d^2y}{dx^2}=D(D-1)y$

$\therefore$ ① $\Rightarrow [D(D-1)+D+1]y = z\sin z$

$(D^2+1)y = z\sin z$

$\therefore m^2+1=0 \quad \therefore m=\pm i \quad \therefore C.F = c_1\cos z + c_2 \sin z$

$= c_1 \cos \log x + c_2 \sin \log x$

$P.I = \frac{1}{D^2+1} z\sin z = \frac{1}{D^2+1} I.P \text{ of } e^{iz} z = I.P. \frac{1}{D^2+1} z e^{iz}$

$= I.P\ e^{iz}. \frac{1}{(D+i)^2+1} z = I.P\ e^{iz}. \frac{1}{D^2+2Di-1+1} z = I.P\ e^{iz}. \frac{1}{D^2+2Di} z$

$= I.P\ e^{iz}. \frac{1}{2Di\left[1+\frac{D^2}{2Di}\right]} z \quad = I.P\ e^{iz}. \frac{1}{2Di\left[1+\frac{D}{2i}\right]} z$

$= I.P\ e^{iz} \frac{1}{2Di}\left[1-\frac{iD}{2}\right]^{-1} z = I.P e^{iz}. \frac{1}{2Di}\left\{z+\frac{i}{2}\right\}$

$= I.P\ e^{iz}. \frac{1}{2i} \frac{1}{D}\left\{z+\frac{i}{2}\right\} = I.P e^{iz}. \frac{1}{2i}\int z+\frac{i}{2}\,dz$

$= I.P\ e^{iz} \frac{-i}{2}\left\{\frac{z^2}{2}+\frac{iz}{2}\right\} = I.P\ e^{iz}\left\{\frac{iz^2}{4}+\frac{z}{4}\right\}$

$= I.P\left\{\cos z + i\sin z\right\}\left\{\frac{iz^2}{4}+\frac{z}{4}\right\} = \frac{-z^2}{4}\cos z + \frac{z}{4}\sin z$

$\therefore$ solution $y = c_1 \cos \log x + c_2 \sin \log x - \frac{1}{4}(\log x)^2 \cos \log x + \frac{1}{4} \log x \sin(\log x)$

12 (a) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}$

$= \int_0^t \cos a(t-u) \sin bu\, du$

$= \frac{1}{2}\int_0^t \cos\left[at-(a-b)u\right] + \cos\left[at-(a+b)u\right] du$

$= \frac{a \sin at - b \sin bt}{a^2-b^2}$

(ii) $L\left\{\int_0^t \frac{e^t \sin t}{t} dt\right\}$

$L\{\sin t\} = \frac{1}{s^2+1} \qquad \therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1}s\Big]_s^\infty = \cot^{-1}s$

$\therefore L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1)$

$\therefore L\left\{\int_0^t e^t \frac{\sin t}{t}\right\} = \frac{1}{s}\cot^{-1}(s-1)$

12 (b) $y'' - 3y' + 2y = 4t + e^{3t}$ given $y(0)=1$ & $y'(0)=-1$

Take Laplace transform on both sides.

$$L\{y'' - 3y' + 2y\} = L\{4t + e^{3t}\}$$

$$s^2\bar{y} - sy(0) - y'(0) - 3\{s\bar{y} - y(0)\} + 2\bar{y} = \frac{4}{s^2} + \frac{1}{s-3}$$

$$(s^2 - 3s + 2)\bar{y} - s + 4 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\therefore \bar{y}(s^2 - 3s + 2) = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$\therefore \bar{y} = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)} = \frac{3}{s} + \frac{2}{s^2} - \frac{1}{2(s-1)} - \frac{2}{s-2} + \frac{1}{2(s-3)}$$

$$\therefore y = L^{-1}\left\{\frac{3}{s} + \frac{2}{s^2} - \frac{1}{2(s-1)} - \frac{2}{(s-2)} + \frac{1}{2(s-3)}\right\}$$

$$y = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

13 a (i) $\nabla^2(r^n) = \nabla\cdot(\nabla r^n)$

Now $\nabla r^n = i\frac{\partial}{\partial x}r^n + j\frac{\partial}{\partial y}r^n + k\frac{\partial}{\partial z}r^n$

$$= i\,nr^{n-1}\frac{\partial r}{\partial x} + j\,nr^{n-1}\frac{\partial r}{\partial y} + k\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right) = nr^{n-2}\vec{r}$$

$$\therefore \nabla^2(r^n) = \nabla\cdot(nr^{n-2}\vec{r}) = n\nabla\cdot(r^{n-2}\vec{r})$$

$$= n\left[\nabla r^{n-2}\cdot\vec{r} + r^{n-2}(\nabla\cdot\vec{r})\right] \qquad \left[\because \nabla\cdot(\phi\vec{A}) = \nabla\phi\cdot\vec{A} + \phi(\nabla\cdot\vec{A})\right]$$

$$= n\left[(n-2)r^{n-3}\frac{\vec{r}\cdot\vec{r}}{r} + r^{n-2}3\right]$$

$$= n\left[(n-2)r^{n-4}\vec{r}\cdot\vec{r} + 3r^{n-2}\right] = n\left[(n-2)r^{n-4}r^2 + 3r^{n-2}\right]$$

$$= nr^{n-2}[n-2+3] = n(n+1)r^{n-2}$$

13 a (ii) $\vec{A} = 3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k$

$$\nabla \cdot \vec{A} = \operatorname{div} \vec{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0$$

$\therefore \vec{A}$ is solenoidal

$\vec{B} = 2xye^z i + x^2e^z j + x^2ye^z k$

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xye^z & x^2e^z & x^2ye^z \end{vmatrix}$$

$$= i\left[x^2e^z - x^2e^z\right] - j\left[2xye^z - 2xye^z\right] + k\left[2xe^z - 2xe^z\right]$$

$$= \vec{0}$$

$\therefore \vec{B}$ is irrotational

b (i) $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x$

$$\nabla\phi_1 = i\frac{\partial}{\partial x}\phi_1 + j\frac{\partial}{\partial y}\phi_1 + k\frac{\partial}{\partial z}\phi_1$$

$$= i\left[2\lambda x - (\lambda+2)\right] + j\left[\mu z\right] + k\left[\mu y\right]$$

$$(\nabla\phi_1)_{(1,-1,2)} = i\left[2\lambda - \lambda - 2\right] + 2\mu j + \mu k$$

$$= i(\lambda - 2) + 2\mu j + \mu k$$

$\phi_2 = 4x^2y + z^3$

$$\nabla\phi_2 = i\frac{\partial}{\partial x}(4x^2y + z^3) + j\frac{\partial}{\partial y}(4x^2y + z^3) + k\frac{\partial}{\partial z}(4x^2y+z^3)$$

$$= i \cdot 8xy + j\,4x^2 + k\,3z^2$$

$$(\nabla\phi_2)_{(1,-1,2)} = -8i + 4j + 12k$$

Given $(\nabla\phi_1)_{(1,-1,2)} \cdot (\nabla\phi_2)_{(1,-1,2)} = 0 \quad (\because \vec{a}\cdot\vec{b} = 0)$

$\therefore -8(\lambda - 2) + 8\mu + 12\mu = 0$

$-8\lambda + 16 + 20\mu = 0$

$\therefore \lambda = -\frac{1}{2}$ and $\mu = -1$

13 b(ii) $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = \operatorname{div}\left(r^{-3}\vec{r}\right)$

Now $\operatorname{div}(\phi\vec{A}) = \phi(\operatorname{div}\vec{A}) + \vec{A}\cdot \operatorname{grad}\phi$

$\therefore \operatorname{div}(r^{-3}\vec{r}) = r^{-3}(\operatorname{div}\vec{r}) + \vec{r}\cdot\operatorname{grad} r^{-3}$ —— ①

Now $\operatorname{div}\vec{r} = \nabla\cdot\vec{r} = \nabla\cdot(xi+yj+zk)$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = \underline{\underline{3}}$

Now $\operatorname{grad} r^{-3} = \nabla(r^{-3}) = i\frac{\partial}{\partial x}(r^{-3}) + j\frac{\partial}{\partial y}(r^{-3}) + k\frac{\partial}{\partial z}(r^{-3})$

$= i\left[-3r^{-4}\, i\frac{\partial r}{\partial x}\right] + -3r^{-4} j\frac{\partial r}{\partial y} + -3r^{-4}k\frac{\partial r}{\partial z}\Big]$

$= -3r^{-4}\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right] = -3r^{-5}\vec{r}$

$\therefore$ Substituting in ①

$\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = r^{-3}\cdot 3 + \vec{r}\cdot -3r^{-5}\vec{r}$

$= 3r^{-3} - 3r^{-5}r^2 = 3r^{-3} - 3r^{-3} = \underline{\underline{0}}$

14(a) $\iint_S \vec{F}\cdot n\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$

Now $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$

$= \underline{\underline{4-4y+2z}}$

$\iiint \nabla\cdot F\, dv = \iiint 4-4y+2z\; dz\, dy\, dx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 4-4y+2z\; dz\, dy\, dx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+9)\, dy\, dx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy\, dx = \int_{-2}^{2} 42\sqrt{4-x^2}\, dx = 84\int_0^2 \sqrt{4-x^2}\, dx$

$= 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = \underline{\underline{84\pi}}$

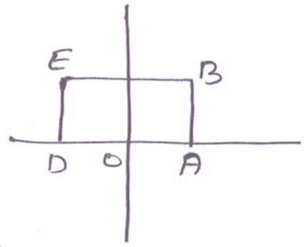
14 (b)

$$\oint_c F \cdot dr = \oint_c [(x^2-y^2)i + 2xy j] \cdot [dx i + dy j]$$

$$= \oint_c (x^2-y^2)\,dx + \oint_c 2xy\,dy = \oint_c (x^2-y^2)\,dx + 2xy\,dy$$

Now the curve C consist of 4 lines AB, BE, ED & DA

Along AB, $x = a \therefore dx = 0$ y varies from 0 to b



$$\therefore \int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy = ab^2$$

Along BE, $y = b \therefore dy = 0$ x varies from a to $-a$

$$\int_{BE} (x^2-y^2)\,dx + 2xy\,dy = \int_a^{-a} (x^2-b^2)\,dx = -2\frac{a^3}{3} + 2ab^2$$

Along ED, $x = -a \therefore dx = 0$ y varies from b to 0

$$\therefore \int_{ED} (x^2-y^2)\,dx + 2xy\,dy = \int_b^0 -2ay\,dy = ab^2$$

Along DA, $y = 0 \therefore dy = 0$ x varies from $-a$ to a

$$\int_{DA} (x^2-y^2)\,dx + 2xy\,dy = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3}$$

$$\therefore \oint_c (x^2-y^2)\,dx + 2xy\,dy = ab^2 - \frac{2a^3}{3} + 2ab^2 + ab^2 + \frac{2a^3}{3} = \underline{4ab^2}$$

Now curl $F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = i(0) - j(0) + k(2y+2y)$

$$= 4yk$$

For the surface S, $\hat{n} = k$

$$\therefore \text{curl} F \cdot n = 4yk \cdot k = 4y$$

$$\therefore \iint_S \text{curl} F \cdot n\,dx = \int_0^b \int_{-a}^{a} 4y\,dx\,dy = \int_0^b (4yx)_{-a}^{a}\,dy = 8a\int_0^b y\,dy$$

$$= 8a\left(\frac{y^2}{2}\right)_0^b$$

$$= 8a\frac{b^2}{2} = \underline{4ab^2}$$

Hence the stoke's theorem is verified.